SOME PROPERTIES OF INSCRIBED QUADRILATERALS.

by

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§ 1. INTRODUCTION.

It is the purpose of this paper to group together the known properties of inscribed quadrilaterals, together with a brief history of each property wherever possible and a proof, or in case the proof is simple, a brief indication of the method of proof. In the collection of material reference has been made only to those sources in the Library of the University of Kansas. The quadrilateral has been considered both as a simple four-line figure and in its complete sense. * These quadrilaterals have been inscribed in circles and conics. The plan has been followed of considering separately the circle and the conic, but cognizance is taken of the fact that a circle is projectively equivalent to a conic. It was felt best to make this division in order to exhibit more clearly the historical development.

* Cf. Veblen and Young. Projective Geometry. (Vol. I, P. 44.

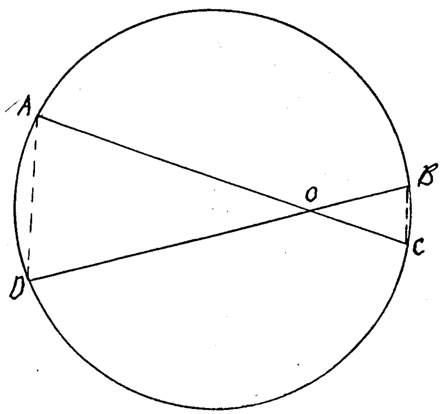
§ 2. Q U A D R I L A T E R A L S
I N S C R I B E D I N C I R C L E S .

<u>FIRST PROPERTY</u>. WHEN A QUADRILATERAL IS INSCRIBED IN A CIRCLE AND THE DIAGONALS INTERSECT IN A POINT, THE PRODUCT OF THE SEGMENTS OF THE ONE DIAGONAL IS EQUAL TO THE PRODUCT OF THE SEGMENTS OF THE OTHER.

The proof of this property follows from the following theorem;-

<u>If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other</u>.

Euclid gives a proof for this theorem in Bk. III, 35, but the proof given here is one usually found in a modern geometry.



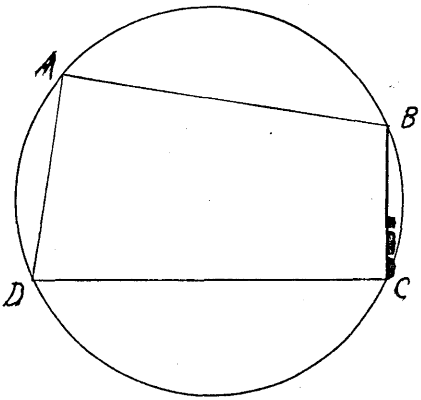
Given the chords AC and BD intersecting at O.

To prove $OA \cdot OC = CB \cdot OD$?

<u>Proof</u>. Draw AD and BC. Triangles AOD and BOC are similar because similar corresponding angles are equal. It follows that $\frac{OA}{OB} = \frac{OD}{OC}$; from which $OA \cdot OC = OB \cdot OD$.

HISTORICAL NOTE. This was one of the earliest theorems applying to the quadrilateral inscribed in the circle and was given by Eudemus * in his History of Geometry, so that we are certain that it was known prior to 300 B.C.

SECOND PROPERTY. THE OPPOSITE ANGLES OF QUADRILATERALS INSCRIBED IN CIRCLES ARE EQUAL TO TWO RIGHT ANGLES.



Euclid gives a long and rigorous proof but the truth of the property is evident when we note that the angles at B and D are together measured by one half the sum of the arcs ADC and ABC, which sum is equal to a complete circumference.

HISTORICAL NOTE. This property is probably the first theorem that deals directly with the inscribed quadrilateral. It is found in Euclid III, 22.**

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* Allman, Greek Geometry from Thales to Euclid, P. 111.

** Ed. Heath. Vol. II, P. 51.

THIRD PROPERTY. IN ORDER THAT A GIVEN QUADRILATERAL MAY BE CIRCUMSCRIBED BY A CIRCLE IT IS NECESSARY THAT THE SUM OF THE OPPOSITE ANGLES OF THE QUADRILATERAL BE EQUAL TO TWO RIGHT ANGLES.

This property follows as the converse of the SECOND PROPERTY and may be proved* by passing a circle thru the vertices of the triangle ABC (of the preceding figure) and then proving, by reductio ad absurdum, that the circle passes thru the fourth point D.

FOURTH PROPERTY. A METHOD OF INSCRIBING IN A GIVEN CIRCLE A QUADRILATERAL EQUIANGULAR TO A GIVEN QUADRILATERAL.

METHOD. It is necessary that in the given quadrilateral, the sum of the opposite angles be equal to two right angles before an equiangular quadrilateral may be inscribed in the given circle. In case the opposite angles are equal to two right angles the problem may be solved by dividing the quadrilateral into two triangles and applying the method of Euclid IV, 2. That is, of inscribing a triangle equiangular to one of the triangles into which the quadrilateral was divided and then forming on the side corresponding to the diagonal, as a base, another

* Todhunter's Euclid. P. 277.

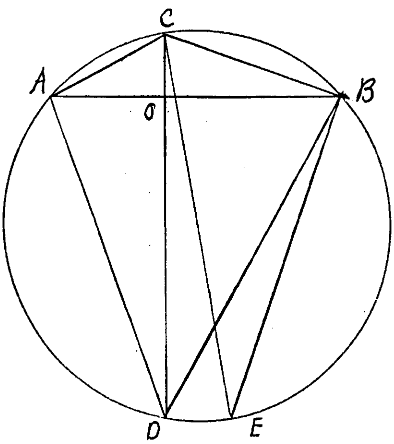
triangle equiangular to the remaining triangle. There are an infinite* number of solutions in which the inscribed quadrilateral will not be of the same form as the given quadrilateral.

<u>FIFTH PROPERTY</u>. IN AN INSCRIBED QUADRILATERAL WHOSE DIAGONALS ARE PERPENDICULAR, THE SUM OF THE SQUARES OF THE FOUR SEGMENTS OF THE DIAGONALS IS EQUAL TO THE SQUARE OF THE DIAMETER OF THE CIRCUMSCRIBING CIRCLE.

This property of the quadrilateral is derived from the following theorem;-

<u>If two chords, AB and CO of a circle intersect at right angles in a point O, the sum of the squares on AO, BO, CO, DO is equal to the square on the diameter.</u>

<u>Proof</u>.** Draw diameter CE and join BE, AD, AC, AND CB.



Angle CAO = Angle CEB. rt. angle COA = rt. angle CBE. Therefore angle ACO = Angle ECB and arcs AD and EB areequal and chords AD and BE are equal. From right triangles-

$$\overline{AO}^2 + \overline{DO}^2 = \overline{AD}^2 = \overline{BE}^2$$
$$\overline{CO}^2 + \overline{BO}^2 = \overline{BC}^2$$

$$\overline{AO}^2 + \overline{DO}^2 + \overline{CO}^2 + \overline{BO}^2 = \overline{BC}^2 + \overline{BE}^2 = \overline{CE}^2$$

* Heath's Euclid, II, 94.

** Heath's Euclid, Vol. II, P. 65.

HISTORICAL NOTE. This theorem is from the "Liber Assumptorum" and has reached us thru the Arabic* It is attributed to Archimedes, but Heath**feels that perhaps much of this is erroneous.

Simpson adds to Bk VI of Euclid as propositions B, C and D, several important propositions which are proved by Euclid VI, 16. These give rise to the SIXTH, SEVENTH and EIGHTH PROPERTIES.

<u>SIXTH PROPERTY</u>. GIVEN AN INSCRIBED QUADRILATERAL ABCD SUCH THAT THE ANGLE AT A IS BISECTED BY ITS DIAGONAL; THE DIAGONALS INTERSECTING AT THE POINT E, IT FOLLOWS THAT:-

$$AD \cdot AB = DE \cdot EB + \overline{EA}^2$$

This property is derived from the following;-

<u>Proposition B</u>. "<u>If the greater angle of a triangle be bisected by a straight line which cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square on the straight line which bisects the angle.</u>"

Given triangle ABD with AE bisecting angle DAB.

To prove $AD \cdot AB = DE \cdot EB + \overline{EA}^2$

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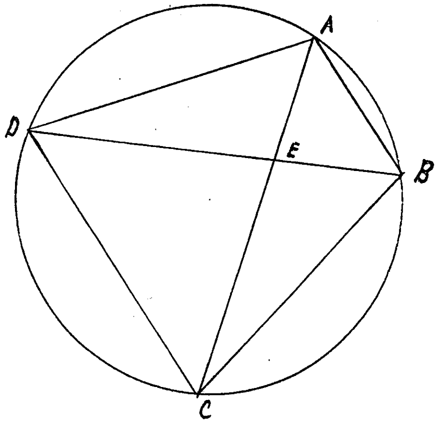
* Archimedes, Ed. Heiberg, II. Pp. 439 - 40.

** Heath's Euclid. Vol. II P. 64.

Proof. Describe circle DAB about the triangle and produce AE to meet the circle at C.



Triangles DAE and CAB are similar and $\frac{DA}{AE} = \frac{AC}{AB}$ or

$DA \cdot AB = AE \cdot AC.$

But $AE \cdot AC = AE \cdot EC + \overline{AE}^2$

and $AE \cdot EC = DE \cdot EB$

Therefore $AD \cdot AB = DE \cdot EB + \overline{EA}^2.$

SEVENTH PROPERTY. IF IN AN INSCRIBED QUADRILATERAL A LINE BE DROPPED FROM ANY VERTEX PERPENDICULAR TO THE OPPOSITE DIAGONAL; THE PRODUCT OF THE TWO SIDES AT THAT VERTEX WILL EQUAL THE PRODUCT OF THE PERPENDICULAR DISTANCE AND THE DIAMETER OF THE CIRCUMSCRIBING CIRCLE.

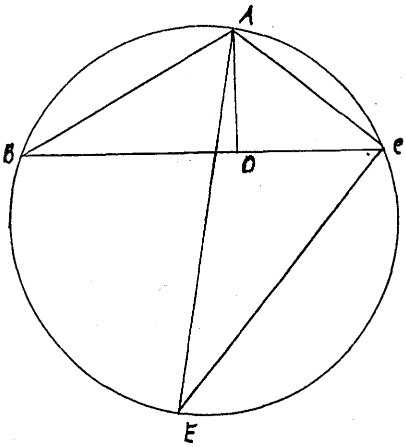
This property is derived from the following;-

Proposition C. If from any angle of a triangle a straight line be drawn perpendicular to the opposite side, the rectangle contained by the other two sides of the triangle is equal to the rectangle obtained by the perpendicular and the diameter of the circle circumscribed about the triangle.

Given the inscribed triangle ABC and AD perpendicular from A to BC, also the diameter AE.

To prove $BA \cdot AC = AD \cdot AE.$

<u>Proof</u>. Triangles ABD and AEC are similar and from this



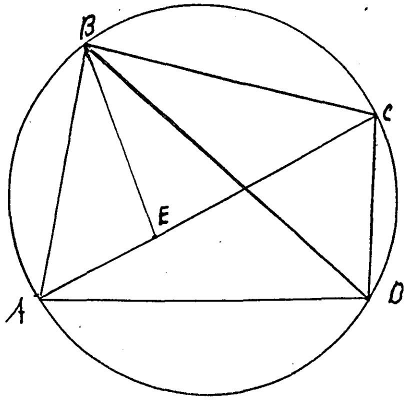
$\frac{BA}{AD} = \frac{EA}{AC}$ or $BA \cdot AC = AD \cdot AE$.

<u>EIGHTH PROPERTY</u>. "THE RECTANGLE CONTAINED BY THE DIAGONALS OF ANY QUADRILATERAL INSCRIBED IN A CIRCLE IS EQUAL TO THE SUM OF THE RECTANGLES CONTAINED BY THE PAIRS OF OPPOSITE SIDES."

This property is itself proposition D.

Given the quadrilateral ABCD inscribed in a circle.

To prove $AC \cdot BD = AB \cdot CD + AD \cdot BC$.



<u>Proof</u>. Draw BE such that angles ABE and DBC are equal. Add to each the angle EBC and it follows that angles ABD and EBC are equal. Triangles ABD and EBC are similar and

$\frac{AD}{AE} * \frac{EC}{DC}$ OR $BA \cdot DC = AE \cdot BD$ -----(1)

Again angles ABE and DBC are equal and also angles BAE and BDC. Triangles ABE and DBC are similar and

$\frac{BA}{AE} =* \frac{BD}{DC}$ or $BA \cdot DC = AE \cdot BD$ --------(2)

by (1) and (2)

$$AD \cdot CB + BA \cdot DC = BD \cdot EC + BD \cdot AE$$
$$= BD \cdot AC.$$

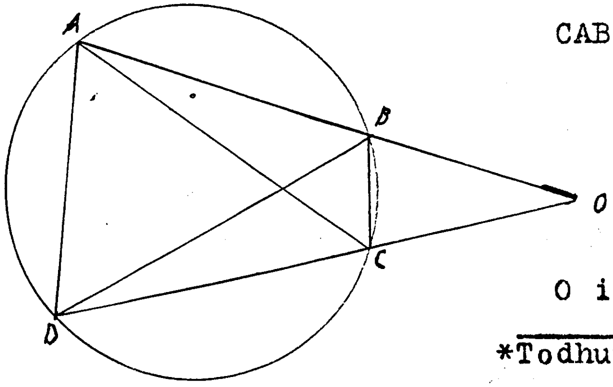
Most of the following properties are exercises usually found in an elementary text book on Plane Geometry.

NINTH PROPERTY. THE STRAIGHT LINES, DRAWN AT RIGHT ANGLES FROM THE MIDDLE POINTS OF THE SIDES OF AN INACRIBED QUADRILATERAL, INTERSECT AT A FIXED POINT. *

This property is obviously truesince the perpendicular bisectors of all chords of a circle intersect in a point.

THENTH PROPERTY. IN THE INSCRIBED QUADRILATERAL ABCD WITH SIDES AB AND CD PRODUCED TO MEET AT O, THE TRIANGLE AOC AND BOC ARE SIMILAR. **

The proof is evident when it is noted that angles



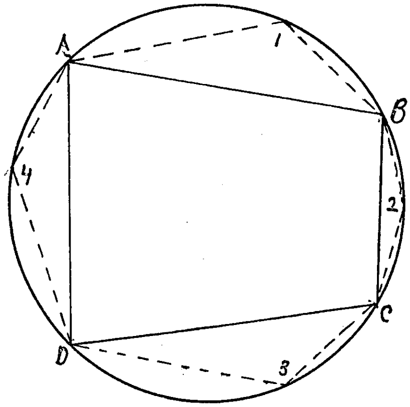
CAB and CDB are measured by the same arc and that angle O is in common.

*Todhunter's Euclid Ex. 163.

**Todhunter's Euclid Ex. 208.

ELEVENTH PROPERTY. IF A QUADRILATERAL IS INSCRIBED IN A CIRCLE, THE SUM OF THE ANGLES IN THE FOUR SEGMENTS OF THE CIRCLE EXTERIOR TO THE QUADRILATERAL IS EQUAL TO SIX RIGHT ANGLES.

To prove $1 + 2 + 3 + 4 =$ 6 rt. angles.



1 = 2rt. angles - ADB

2 = 2rt. angles - BDC

3 = 2rt. angles - DBC

4 = 2rt. angles - ABD

1 + 2 + 3 + 4 = 8 rt. angles - (ADB+BDC+DBC+ABD)

= 8rt. angles - (ADC+ABC)

= 8 rt. angles - 2 rt. angles =

= 6 rt. angles.

TWELFTH PROPERTY. THE OPPOSITE SIDES OF THE INSCRIBED QUADRILATERAL ABCD, PRODUCED, MEET AT P AND Q. IF OP AND OQ BISECT THE ANGLES AT P AND Q RESEECTIVELY THE LINES OP AND OQ ARE PERPENDICULAR TO EACH OTHER.*

To prove QO perpendicular to PO.

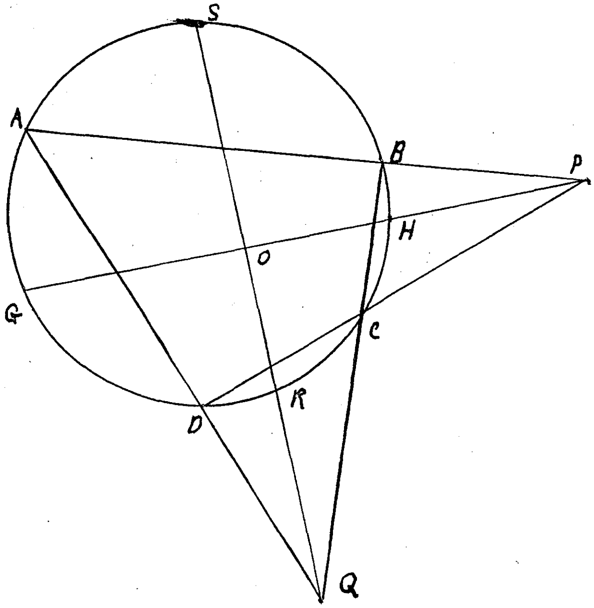
Proof. $\frac{1}{2}(\overset{\frown}{AS} - \overset{\frown}{DR}) = \frac{1}{2}(\overset{\frown}{SB} - \overset{\frown}{RC})$

$\underline{\frac{1}{2}(\overset{\frown}{AG} - \overset{\frown}{BH}) = \frac{1}{2}(\overset{\frown}{GD} - \overset{\frown}{HC})}$

$\overset{\frown}{AG} + \overset{\frown}{AS} + \overset{\frown}{RC} + \overset{\frown}{HC} = \overset{\frown}{SB} + \overset{\frown}{BH} + \overset{\frown}{RD} + \overset{\frown}{DG}$

* Todhunter's Euclid, Ex. 216.

Angle ROH is measured by $\frac{1}{2}(\overset{\frown}{AG} + \overset{\frown}{AS} + \overset{\frown}{RC} + \overset{\frown}{HC})$



Angle HOS is measured by $\frac{1}{2}(\overset{\frown}{SB}+\overset{\frown}{BH}+ \overset{\frown}{RD}+\overset{\frown}{DG})$

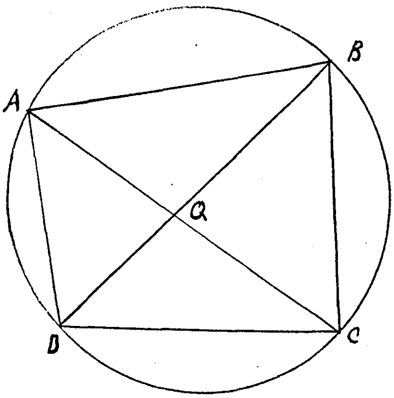
Therefore angle ROH = angle HOS.

But angle ROH + ANGLE HOS = 2 rt. angles.

Therefore the angle ROH = 1 rt. angle and PO is perpendicular to OQ.

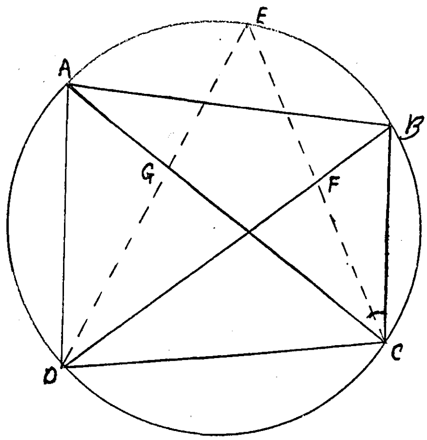
<u>THIRTEENTH PROPERTY.</u> THE DIAGONALS OF ANY QUADRILATERAL INSCRIBED IN A CIRCLE DIVIDE THE QUADRILATERAL INTO FOUR TRIANGLES WHICH ARE SIMILAR, TWO AND TWO.*

The property is true when it is noted that in triangles ADQ and BCQ, angles DAQ and CBQ are measured by the same arc as are angles ADQ and QCB.



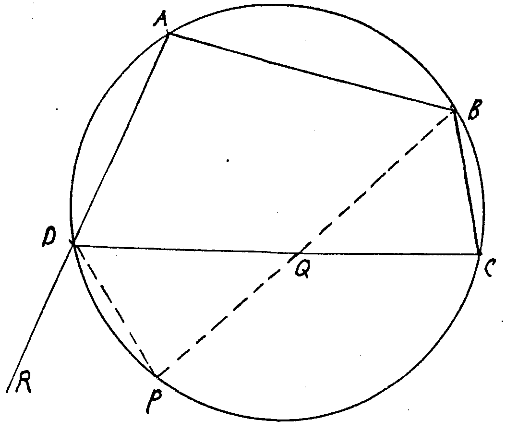
* Todhunter's Euclid, Ex. 299.

FOURTEENTH PROPERTY. IN THE INSCRIBED QUADRILATERAL ABCD THE STRAIGHT LINES WHICH BISECT THE ANGLES ACB AND ABD CUT BD AND AC AT F AND G RESPECTIVELY. IT FOLLOWS THAT EF:EG = ED:EC.



This property is proved from the relations of the two similar triangles DEF AND GEC.

FIFTEENTH PROPERTY. THE STRAIGHT LINES BISECTING ANY ANGLE OF A QUADRILATERAL AND THE OPPOSITE EXTERIOR ANGLE, MEET OF THE CIRCUMFERENCE OF THE CIRCLE.*



Given in the inscribed quadrilateral ABC, the angles ABC and RDC bisected by BP and DP respectively.

To prove that DP and PB intersect on a circle ABC.

Proof. Let BQ, the bisector of angle ABC, intersect the circle at P and join P to D.

* Todhunter's Euclid, Ex. 223.

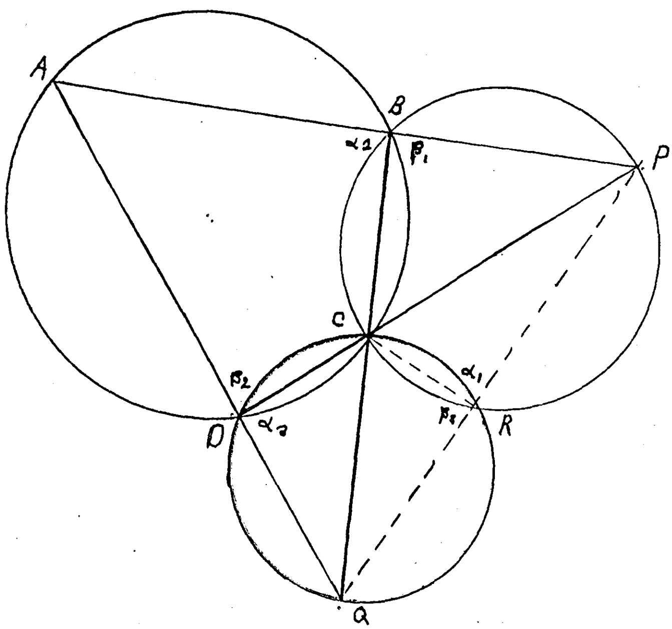
Angles RDC and ABC are equal.

Angles PDC and PBC are measured by the same arc and are therefore equal.

But the angle PBC is one-half the angle ABC and therefore the angle PDC is one-half of the angle RDC. Thus the initial conditions are satisfied if the intersection of the bisectors PB and DP lies on the circle.

<u>SIXTEENTH PROPERTY</u>. THE OPPOSITE SIDES OF A QUADRILATERAL INSCRIBED IN A CIRCLE ARE PRODUCED TO MEET AT P AND Q; AND ABOUT THE TRIANGLES SO FORMED WITHOUT THE QUADRILATERAL, CIRCLES ARE DESCRIBED MEETING AGAIN AT R. IT FOLLOWS THAT P, R, AND Q ARE ON A STRAIGHT LINE.*

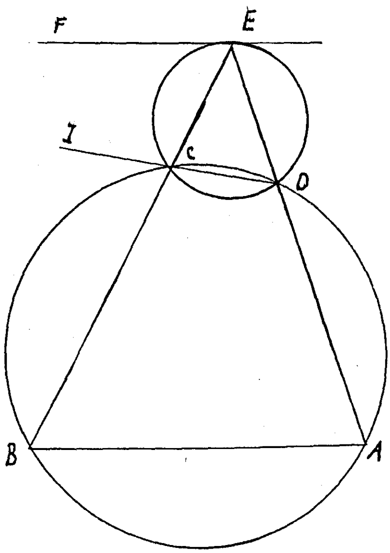
<u>Proof</u>. To prove that P, Q and R all lie on the same straight line it is only necessary to show that the angle α_1 is supplementary to the angle β_3. This is at once seen to be true if it is noted that in the figure each α_i is supplementary to each β_i.



* Todhunter's Euclid. Ex. 309.

SEVENTEENTH PROPERTY. THE QUADRILATERAL ABCD IS INSCRIBED IN A CIRCLE AND AD, BC ARE PRODUCED TO MEET AT E. IT FOLLOWS THAT THE CIRCLE DESCRIBED ABOUT THE TRIANGLE ECD WILL HAVE THE TANGENT AT E PARALLEL TO AB.*

To prove the tangent EF parallel to the side AB it is necessary to show that angles HAB and DEF are equal.



Angles HAB, ICE and CDE + DEC are equal. But angles CDE and CEF are equal. Therefore angles HAB, DEC + CEF, AND DEF are equal.

EIGHTEENTH PROPERTY. IF A QUADRILATERAL BE INSCRIBED IN A CIRCLE, AND A STRAIGHT LINE BE DRAWN MAKING EQUAL ANGLES WITH ONE PAIR OF OPPOSITE SIDES, IT WILL MAKE EQUAL ANGLES WITH THE OTHER PAIR.**

Given the inscribed quadrilateral ABCD cut by a transversal QN so that angles IOC and DNO are equal.

To prove angles BRS and RSC are equal.

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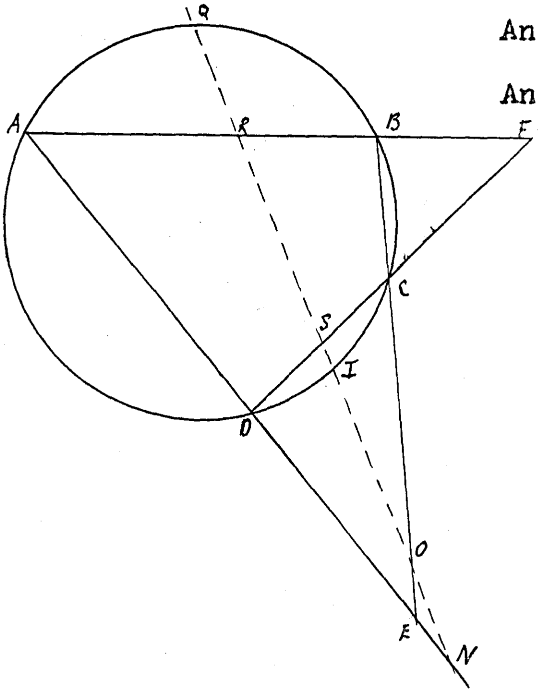
* Todhunter's Euclid, Ex. 299.

**Todhunter's Euclid, Ex. 217.

(1)--- angle IOC = $\frac{1}{2}(\widehat{QB} - \widehat{IC})$ = angle DNO = $\frac{1}{2}(\widehat{QA} - \widehat{DI})$



Angle CSR = $\frac{1}{2}(\widehat{QB} + \widehat{BC} + \widehat{DI})$

Angle BRS = $\frac{1}{2}(\widehat{IC} + \widehat{CB} + \widehat{AQ})$

If angles IOC and DNO are equal the following equation must be satisfied;

(2) $\widehat{QB}+\widehat{BC}+\widehat{DI} = \widehat{IC}+\widehat{CB}+\widehat{AO}$

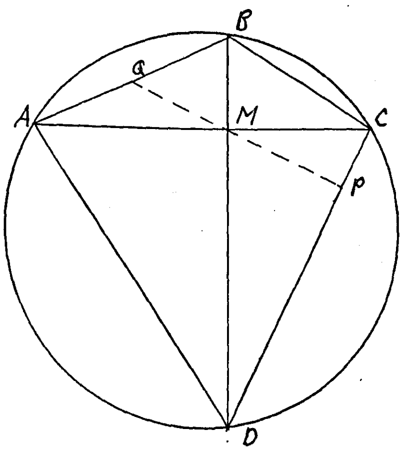
From (1) $\widehat{AQ} = \widehat{QB}-\widehat{IE}+\widehat{DI}$

Substituting in (2) it becomes-

$\widehat{QB} + \widehat{DI} = \widehat{IE} + \widehat{QB} - \widehat{IE} + \widehat{DI}$

which is satisfied.

<u>NINETEENTH PROPERTY</u>. IN THE INSCRIBED QUADRILATERAL ABCD HAVING THE DIAGONALS PERPENDICULAR AT M, THE PERPENDICULAR ON CD FROM M PASSES THRU THE MID-POINT OF THE OPPOSITE SIDE AB.

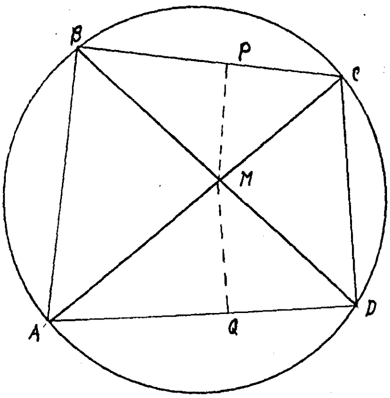


The proof of this property consists in showing by a comparison of angles that in triangle QMB, QM = OB; and that in triangle QMA, QM = QA.

It then follows at once that AQ = QB.

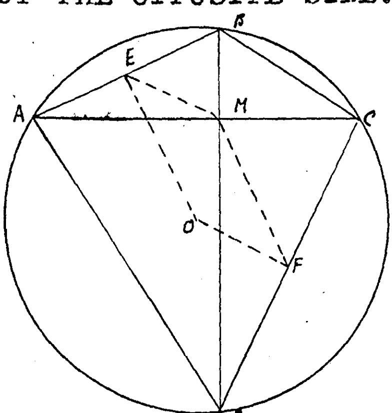
HISTORICAL NOTE. This theorem was discovered by Brahmagupta.*

TWENTIETH PROPERTY. IN THE INSCRIBED QUADRILATERAL ABCD, THE DISTANCES MP AND MQ, FROM THE POINT OF INTERSECTION OF THE DIAGONALS, TO THE OPPOSITE SIDES AB AND CD ARE PROPORTIONAL TO THE TWO SIDES.**



The proof consists in showing that triangle BMC is similar to triangle ADM and therefore their altitudes are proportional to their bases.

TWENTY-FIRST PROPERTY. IN THE QUADRILATERAL INSCRIBED IN A CIRCLE SO THAT THE DIAGONALS ARE PERPENDICULAR, THE DISTANCE OE FROM THE CENTER TO THE SIDE AB EQUALS THE HALF OF THE OPPOSITE SIDE.***



The truth of this theorem is apparent when it is noted that EMFO is a parallelogram and that MF equals CF. From this it follows that OE equals CF which is

* Charles. Apercu Historique, 2nd. Ed. P. 435.
**E. Catalan, Geometrie. P. 154.
*** Sancery. Novelles Annales. 2me, seriet X.

one-half of CD or the opposite side.

TWENTY SECOND PROPERTY. WHERE a AND e ARE THE SIDES AB AND CD AND r IS THE RADIUS OF THE CIRCLE,- $a^2 + e^2 = 4r^2$

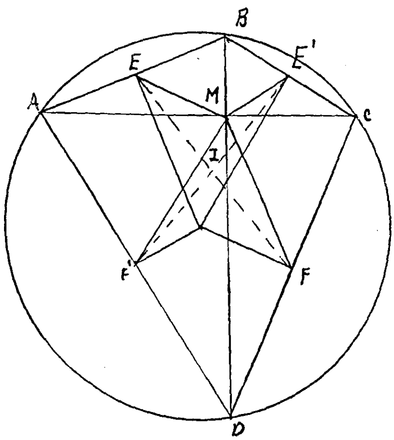
This property follows as a corollary to the TWENTY-FIRST PROPERTY.

TWENTY-THIRD PROPERTY. IF THE INSCRIBED QUADRILATERAL OF THE TWENTY-FIRST PROPERTY BE ROTATED ABOUT THE FIXED POINT,-

I. THE MIDPOINTS OF THE FOUR SIDES DESCRIBE A CIRCUMFERENCE WITH CENTER I, THE MIDPOINT OF MO.

II. EF IS THE DIAMETER OF THE CIRCLE.

III. THE RADIUS R' IS GIVEN BY THE FORMULA:-



$4R'^2 = 2R^2 - \overline{MO}^2$

To prove the first and second parts of this property it is necessary to show that EI = E'I

Proof. Consider the parallelogram MFOE,

$\overline{OE}^2 + \overline{OF}^2 = 2\overline{EI}^2 + 2\overline{MI}^2$

or $\frac{1}{4}(\overline{CD}^2 + \overline{AB}^2) = \frac{1}{2}\overline{EF}^2 + \frac{1}{2}\overline{OM}^2$.

It follows that $\overline{EF}^2 = 2R^2 - \overline{MO}^2$.

Considering parallelogram F'OE'M it may be shown in a similar way that $\overline{E'F'}^2 = 2R^2 - \overline{MO}^2$.

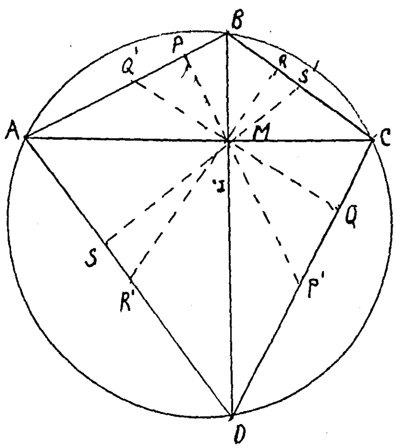
Therefore EF = E'F' and EI = E'I.

The third part follows when it i s noted that

EF = 2R' = E'F'

<u>THWENTY - FOURTH PROPERTY.</u> IN THE INSCRIBED QUADRILATERAL ABCD WITH THE DIAGONALS PERPENDICULAR TO EACH OTHER, THE LINES PERPENDICULAR TO EACH SIDE, THRU THE POINT OF INTERSECTION OF THE DIAGONALS, CUT THE OPPOSITE SIDES IN FOUR POINTS. THE EIGHT POINTS, COMPOSED OF THESE FOUR POINTS AND THE FOUR FEET OF THE PERPENDICULAR LIE ON A CIRCUMFERENCE WHOSE CENTER AND RADIUS DEPEND UNIQUELY ON THE CIRCUMFERENCE ABCD AND THE POSITION OF THE POINT M.

An outline of the proof is as follows;-

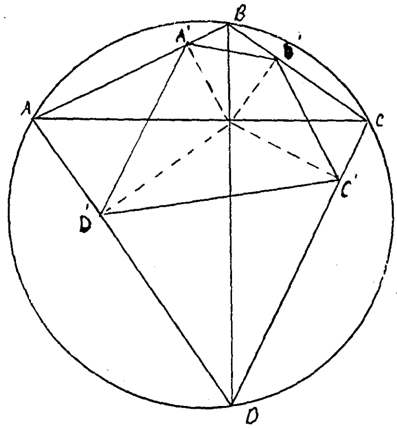


Considering O'P' as a diameter a circle may be passed thru P and Q since the angles at P and Q are right angles.

Also a circle may be passed thru R and S with R'S' as a diameter. It follows that these two circles are the same since their diameters P'S' and P'Q' ARE EQUAL AND BISECT EACH OTHER.

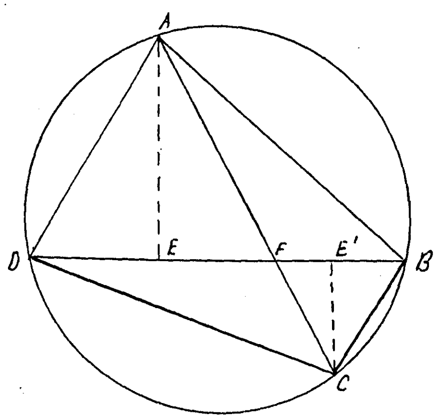
By the TWENTY-THIRD PROPERTY it follows that as the diagonals AC and BD rotate on M as a center, the center I and the radius R' of the circumference of the eight points remain constant.

TWENTY-FIFTH PROPERTY. IN THE INSCRIBED QUADRILATERAL ABCD WITH DIAGONALS PERPENDICULAR TO EACH OTHER; THE FEET OF THE PERPENDICULARS DROPPED UPON THE SIDES FROM THE INTERSECTION OF THE DIAGONALS ARE THE VERTICES OF A QUADRILATERAL INSCRIBABLE.*



This property follows at once from the TWENTY-FOURTH PROPERTY.

TWENTY-SIXTH PROPERTY. IN AN INSCRIBED QUADRILATERAL ABCD, IF F IS THE INTERSECTION OF THE DIAGONALS AC AND BD, THEN- $\frac{AD \cdot AB}{CB \cdot CD} = \frac{AF}{FE}$.



Proof. Draw AE and CE' perpendicular to DB. Then with d as the diameter of the circumscribing circle we may write-

$$\frac{AD \cdot AB}{CB \cdot CD} = \frac{AE \cdot d}{CE' \cdot d} = \frac{AF}{FE}$$

*E. C atalan. Geometry, 137.

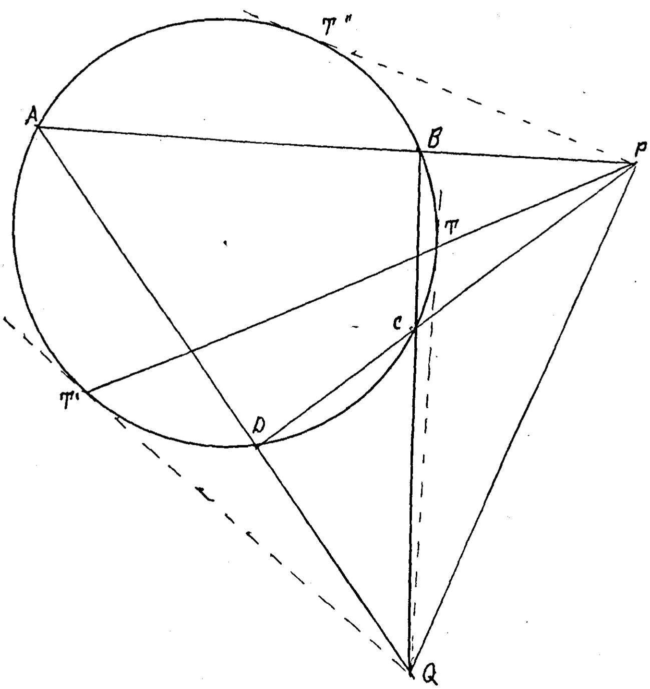
TWENTY-SEVENTH PROPERTY. IN AN INSCRIBED QUADRILATERAL ABCD, - $\frac{AD\ AB + CB\ CD}{BA\ BC + DA\ DC} = \frac{AC}{BD}$ *

Proof from above figure.

$$\frac{AD\ AB + CB\ CD}{AB\ BC + DA\ DC} = \frac{d\ AE + d\ CE'}{d\ BG + d\ DG} = \frac{AF + FE}{BD + DF} = \frac{AC}{BC}$$

TWENTY-EIGHTH PROPERTY. THE OPPOSITE SIDES OF A QUADRILATERAL INSCRIBED IN A CIRCLE WHEN PRODUCED MEET AT P AND Q. THE SQUARE ON PQ IS EQUAL TO THE SUM OF THE SQUARES ON THE TANGENTS FROM P AND Q TO THE CIRCLE.**

To prove, $\overline{PQ}^2 = \overline{QT'}^2 + \overline{PT''}^2$



Proof.*** P and Q are the extremities of the third diagonal.

Now TT', the polar of Q, passes thru P and therefore, by Euclid II,6,

$\overline{QP}^2 - \overline{QT}^2 = PT\ \ PT'$

but PT Pt' is equal to the tangent from P.

Therefore

$\overline{PQ}^2 = \overline{QT'}^2 + \overline{PT''}^2$

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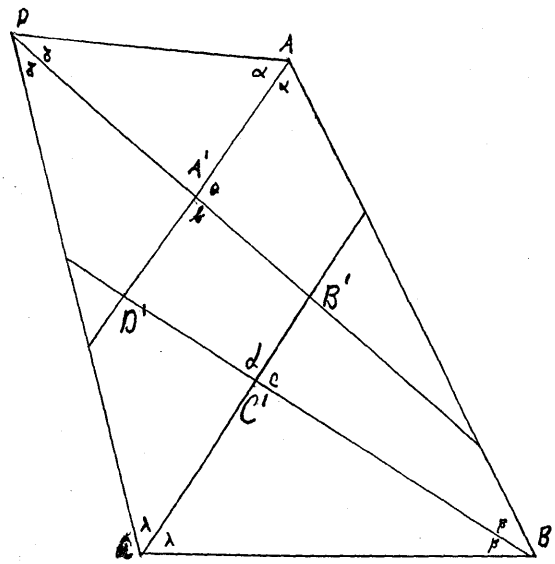
* Chauvent. Elementary Geometry. P. 311.

**Todhunter's Euclid. Ex. 614.

***Mulcahy. Modern Geometry. P. 80.

TWENTY-NINTH PROPERTY. THE FOUR STRAIGHT LINES BISECTING THE ANGLES OF ANY QUADRILATERAL FORM A QUADRILATERAL WHICH CAN BE INSCRIBED IN A CIRCLE.*

Given any quadrilateral ABCD with the bisectors of the four angles forming another quadrilateral A'B'C'D'.



To show that angles b and d are supplementary.

$$a = \delta + \alpha = \tfrac{1}{2}A + \tfrac{1}{2}D$$
$$c = \beta + \lambda = \tfrac{1}{2}B + \tfrac{1}{2}C$$

$$\begin{aligned} a+c &= \tfrac{1}{2}(A+C+B+D) \\ &= \tfrac{1}{2}4 \text{ rt. angles} \\ &= 2 \text{ rt. angles.} \end{aligned}$$

a+b+c+d = 4 rt. angles

2 rt. angles +b+d = 4 rt. angles.

Therefore-

b+d = 2 rt. angles and are supplementary.

THIRTIETH PROPERTY. WITH FOUR GIVEN STRAIGHT LINES TO FORM A QUADRILATERAL INSCRIBABLE IN A CIRCLE.

This problem was propesed by J.W.L. Glaisher and was solved by Cayley in June 1874. His solution is given here.**

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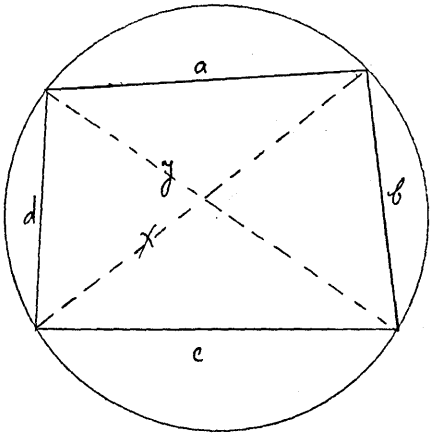
* Todhuntter's Euclid. Ex.

**Cayley, C ollected Math. Papers. Vol. X, P. 579

Solution. Let the sides be in order a, b, c, d, and let the diagonals joining intersection of a, b and c, d be x, and of a, d and c, b be y.



In order to form a quadrilateral inscribable it is necessary that the opposite angles be supplementary to each other.

Let angles subtended by diagonal x, be θ and $\pi - \theta$ and we write-

$$x^2 = b^2 + c^2 + 2b \cos\theta = a^2 + d^2 - 2ad \cos\theta$$

and $(ad + bc)x^2 = ad(b^2 + c^2) + bc(a^2 + d^2) =$

$$(ac + bd)(ab + cd)$$

or $x^2 = (ac + bd)\dfrac{ab + cd}{ad + bc}$, and similarly

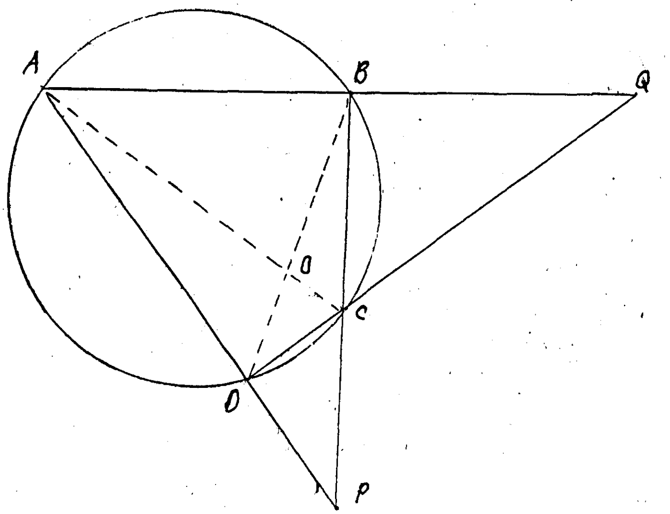
$y^2 = (ac + bd)\dfrac{ad + bc}{ab + cd}$ from which $xy = ac + bd$

Thus is it seen that the above quadrilateral is determined by either of the diagonals.

THIRTY-FIRST PROPERTY. IN THE INSCRIBED QUADRILATERAL, THE INTERSECTION OF THE DIAGONALS AND OF THE OPPOSITE SIDES ARE THREE POINTS SUCH THAT EACH IS THE POLE OF THE LINE JOINING THE OTHER TWO.

To show O, Q and P respectively the poles of PQ, OP and OQ.

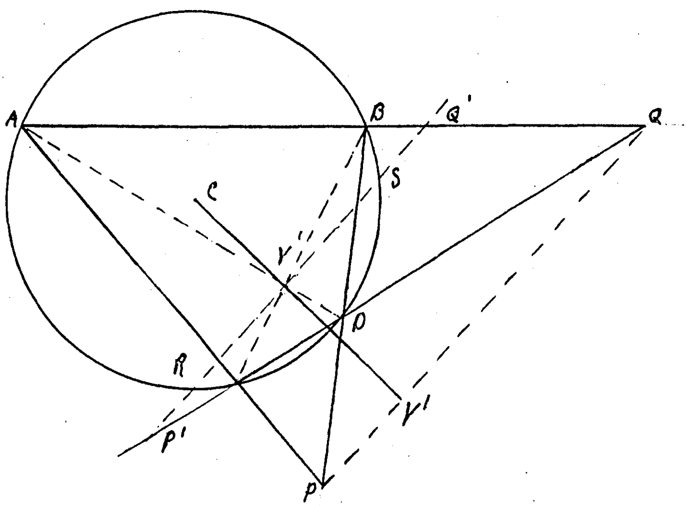
The proof follows at once when it is noted that OR, OQ and PQ are respectively polars of Q, P, and Q, and



that the line joining any two points has for its pole, the intersection of the polars of the points.

<u>THIRTY-SECOND PROPERTY.</u> IF A QUADRILATERAL BE INSCRIBED IN A CIRCLE WITH CENTER C, THE LINE JOINING THE CENTER WITH THE INTERSECTION OF THE DIAGONALS V, IS PERPENDICULAR TO THE THIRD DIAGONAL.*

To show CV' perpendicular to PQ.



<u>Proof.</u> Draw P'Q' thru V parallel to PQ. Then - RV = VS, CR = CS and CV is perpendicular to P'Q' which is perpendicular to PQ.

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* Mulcahy. Modern Geometry. P. 37.

§3. QUADRILATERALS INSCRIBED IN CONICS.

In this section the complete quadrilateral has been considered as a figure composed of four lines, or sides, and their six meets, or vertices. Lines joining two opposite vertices have been defined as diagonals. As it is impossible to pass a non-degenerate conic thru more than four vertices, the complete quadrilateral has been considered as inscribed when four of its vertices lie on the conic.

<u>FIRST PROPERTY</u>. EVERY CONIC SECTION HAS, WITH REFERENCE TO ANY INSCRIBED QUADRILATERAL, THE PROPERTY OF THE FOUR LINE LOCUS.

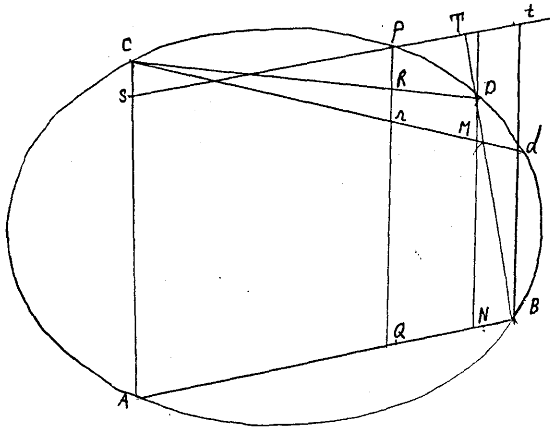
Put in more modern phraseology this property reads-

IN A QUADRILATERAL INSCRIBED IN A CONIC, THE RATIO OF THE PRODUCT OF THE DISTANCES OF ANY POINT OF THE CURVE FROM TWO OPPOSITE SIDES OF THE QUADRILATERAL TO THE PRODUCT OF THE DISTANCES OF THE SAME POINT FROM THE OTHER TWO SIDES IS CONSTANT.

Given the inscribed quadrilateral ABCD, and P any point on the conic. Also the lines PQ, PR, PS, PT meeting the sides at given angles.

To prove $\frac{PQ \cdot PR}{PS \cdot PT} = k.$

NEWTON'S PROOF. Newton first proves this property for the special case where two sides are parallel and he then proceeds to the case where no two sides are parallel. His proof of the latter case is as follows;-



Draw Bd parallel to AC, meeting ST at t and the conic at d. Join Cd cutting PQ in r and draw DM parallel to PQ cutting Cd in M and AB in N. Then it follows by similar triangles and by parallels that-

$$\frac{Bt}{Tt} = \frac{DN}{NB} = \frac{PQ}{Tt} \quad \text{and} \quad \frac{Rr}{AQ} = \frac{DM}{AN} = \frac{Rr}{PS}$$

Therefore $\frac{PQ \cdot Rr}{PS \cdot Tt} = \frac{ND \cdot DM}{AN \cdot NB} = \frac{PQ \cdot Pr}{PS \cdot Pt}$ (This last equality by his proof when two sides are parallel.)

$$\frac{PQ \cdot PR}{PS \cdot PT} = \frac{ND \cdot DM}{AN \cdot NB}$$

He then follows by the statement, "Having thus shown that this last ratio has a constant value $\frac{DN \cdot DM}{AN \cdot NB}$, we see at once that PQ PR will still vary as PS PT if PQ, PR, PS, PT be drawn each at its own constant inclination to AB, CD, AC, BD respectively." In particular it would hold for lines perpendicular to the sides of the quadrilateral and the above property is demonstrated.

HISTORICAL NOTE. Heath says*, "that the words used (by Apollonius in his first preface) indicate clearly that Apollonius did himself possess a complete solution of the problem of the four line locus, and the remarks of Pappus** on the subject tho not friendly to Apollonius, confirm the same inference." However in the extant works of Apollonius no mention whatever is made of this locus.

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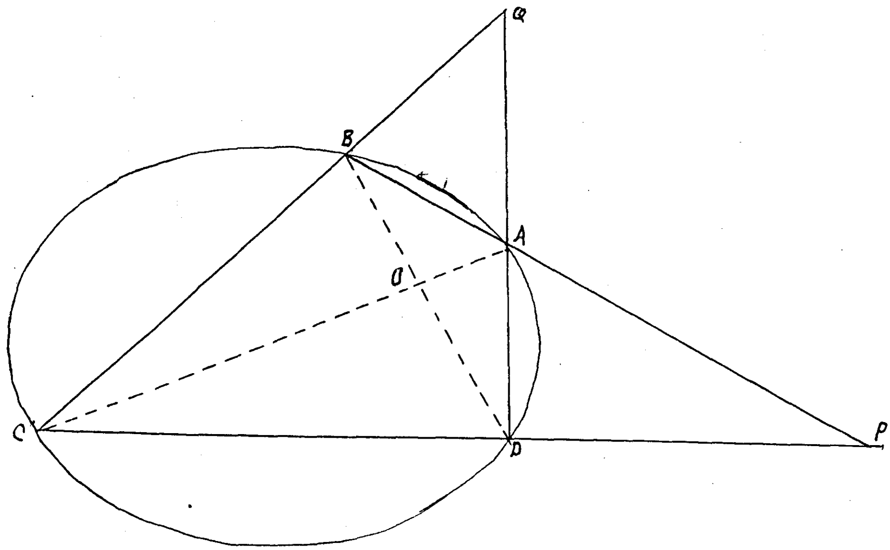
* Apollonius of Perga P,CXXXVIII.

** Pappus:- "Apollonius says in his third book that the locus with respect to three or four lines' had not been completely investigated by Euclid, and in fact neither Apollonius himself nor any one else could have added in the least to what had been proved up to Euclid's time: Apollonius himself is evidence for this fact when he says that the theory of that locus could not be completed without the proposition which he had been obliged to work out for himself."

Collectia Lib. VII, Vol. II, Pp. 677-9. Ed. Hultschi.

This theorem was solved by Descartes, by his new method of coordinates, (Geometria Libb. I, II, 7-16, 24-34. Ed. Schooten 1659.) and completely solved by Newton (Principia, Lib. I, Sect. V. lemma 17-19.) by the elementary geometry of Apollonius. This last solution is the one given here.

<u>SECOND PROPERTY</u>. THE INTERSECTIONS OF OPPOSITE SIDES AND THE DIAGONALS OF A QUADRILATERAL ARE A CONJUGATE TRIAD WITH RESPECT TO EVERY CONIC CIRCUMSCRIBING THE QUADRILATERAL.



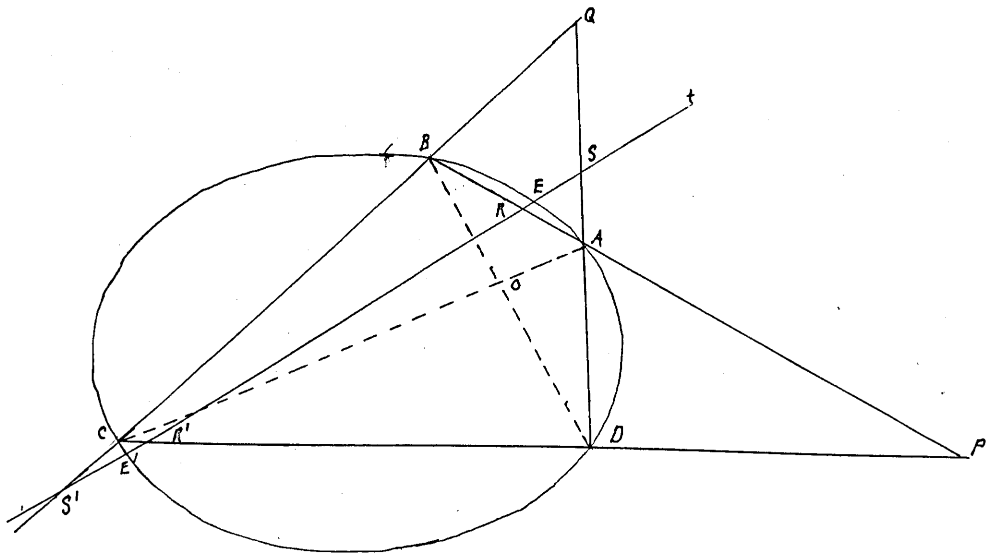
To prove that P, Q and O are a conjugate triad with respect to the conic ABCD.

<u>Proof</u>. This property is seen to be true when it is noted that the line OP and the point Q divide AD and BC harmonically. OP is the polar of Q, OQ is the polar

of P, and therefore Q being the pole of PQ, it follows that the points O, P, Q are a conjugate triad.

HISTORICAL NOTE. This theorem was given by Desargues in his Brouillon Proiect etc., Pp. 188-9.

THIRD PROPERTY. IN THE QUADRILATERAL ABCD INSCRIBED IN A CONIC, THE FOUR SIDES, THE TWO DIAGONALS AC AND BD, AND CONIC MEET EVERY TRANSVERSAL IN FOUR PAIRS OF POINTS IN INVOLUTION.



Let t, and transversal, meet the conic at E and E' and pairs of opposite sides of the quadrilateral at S, S' and R, R'.

To show the pairs of points E, E' ; S, S' ; R, R' are in involution.

Proof. By the property of the "four line locus" we may write $\frac{ER\cdot ER'}{ES\cdot ES'} = \frac{E'R\cdot E'R'}{E'S\cdot E'S'}$

But this is the cross ratio of four points equal to the cross ratio of their four conjugates which is a well known property of the involution.

HISTORICAL NOTE. This is one of the fundamental theorems of Desargues. He first proved it for a circle and then extended it to the general conic by projection, but did not prove it for the general conic.* The theorem seems to have been first stated for the case of three conics, instead of one conic and an inscribed quadrilateral, by Sturm.**

The following may be added as corollaries;-

COROLLARY I. The four sides of a given quadrilateral suffice to determine an involution onany transversal and every conic circumscribing the quadrilateral passes thru an additional couple in such involution.

COROLLARY II. The foci of the involution determined upon any transversal by the sides of the quadrilateral taken in opposite pairs with respect to the quadrilateral are conjugate points with respect to every conic circumscribing it. The polar of a given point F, with respect

* Poudra's Oeuvres de Desargues, I, p. 174 & 193.

** Gergonnes' Annales XVII, p. 180.

to a system of conics thru four given points ABCD, passes thru a fixed point F', which is determined as the second focus of the transversal which touches the conic ABCDF at F'.

Chasles* gives the following as a corollary;-

COROLLARY III. When two quadrilaterals are inscribed in one conic, if three sides of the first cut respectively three sides of the second in three points on a straight line, the point of intersection of the fourth is on the same straight line.

NOTE. Compare this corollary with the SEVENTH PROPERTY.

FOURTH PROPERTY. IN A QUADRILATERAL INSCRIBED IN A CONIC;-

CASE I. THE RECTANGLES UNDER THE SEGMENTS OF THE DIAGONALS ARE TO EACH OTHER AS THE SQUARES OF THE DIAMETERS PARALLEL TO THE DIAGONALS.

CASE II. THE RECTANGLES UNDER THE SEGMENTS OF THE DIAGONALS ARE TO EACH OTHER AS THE SQUARES OF PARALLEL TANGENTS.

HISTORICAL NOTE. Case II was first stated and proved by Apollonius** but his proof is long. Salmon***proves

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* Sections Coniques, p. 90.

** Edition Heath. pp. 95-6.

*** Salmon, Conic Sections, pp. 145-6.

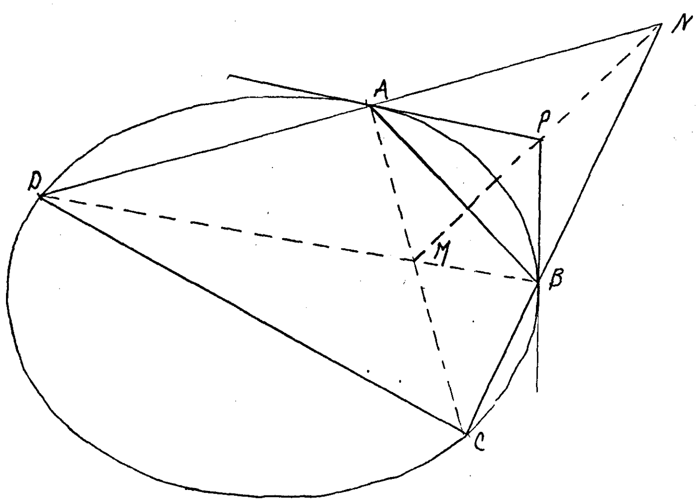
both cases by reference to the theorem;- If thru two fixed points O and O', any parallel lines OR and O'P be drawn, then the ratio of the rectangles $\frac{OR' \cdot OR''}{O'P' \cdot O'P''}$ will be constant, whatever the direction of these lines.

<u>Proof of CASE I.</u> If O' be the center of the curve, then Op' = O'P'' and the quantity O'p' O'P'' becomes the square of the semi-diameter parallel to OR'.

<u>Proof of CASE II.</u> If the line OR be a tangent, then OR' = OR'' and the quantity OR' OR'' becomes the square of the tangent.

<u>FIFTH PROPERTY.</u> IN THE QUADRILATERAL ABCD INSCRIBED IN A CONIC, THE TANGENTS AT A AND B MEET ON THE LINE THRU THE INTERSECTION OF THE OPPOSITE SIDES, AD AND BC, AND THE INTERSECTION OF THE DIAGONALS.*

To prove that tangents AP and PB intersect on line MN.



<u>Proof.</u>

A(PDCB)=B(ADCP)

But this may be written without changing the value

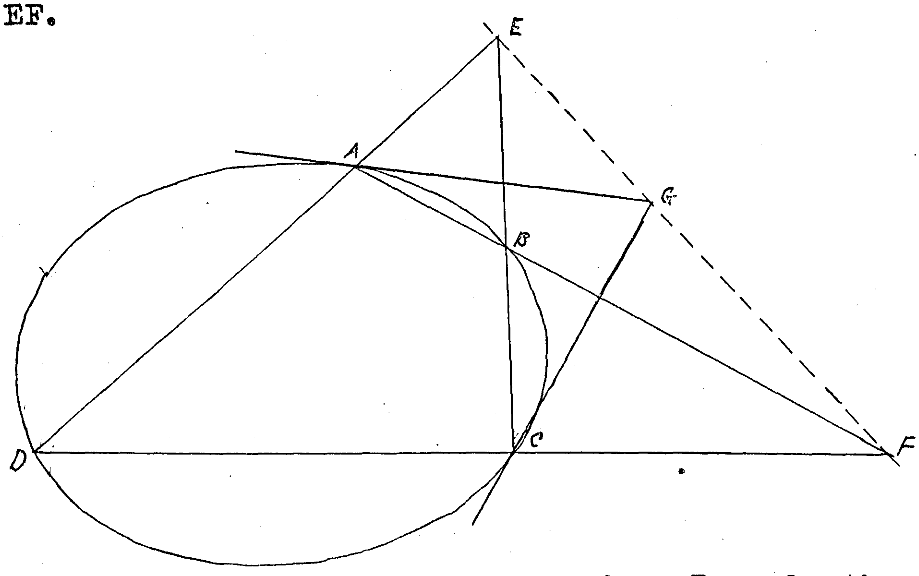
A(PDCB)=B(PCDA).

* Chasles, Traite des Sectione Coniques, pp. 76-7.

Since these two projective pencils have the corresponding line AB in common it follows that the corresponding lines meet in three points P, N, and M of a straight line.

SIXTH PROPERTY. IN THE QUADRILATERAL ABCD INSCRIBED IN A CONIC, THE TWO TANGENTS TO THE QUADRILATERAL AT A AND C, (OPPOSITE VERTICES) MEET IN A POINT ON THE LINE WHICH IS THE JOIN OF THE MEETS OF OPPOSITE SIDES EXTENDED.*

To show that tangents at A and C meet on the line EF.

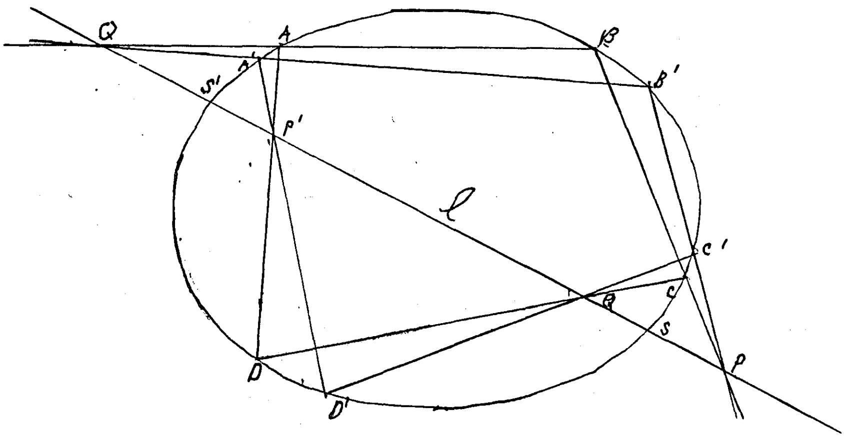


The proof follows at once from Pascals theorem if it is noted that the tangents at A and C are the limiting positions of two sides of a hexagon.

* M. Chasles. Sections Coniques, p. 44.

SEVENTH PROPERTY. IF THREE SIDES OF A QUADRILATERAL INSCRIBED IN A CONIC TURN ABOUT THREE POINTS IN A STRAIGHT LINE, THE FOURTH SIDE TURNS ABOUT A POINT ON THE SAME STRAINGHT LINE.*

Let three sides of the quadrilateral ABCD turn about the fixed points P, Q, and P' on 1.



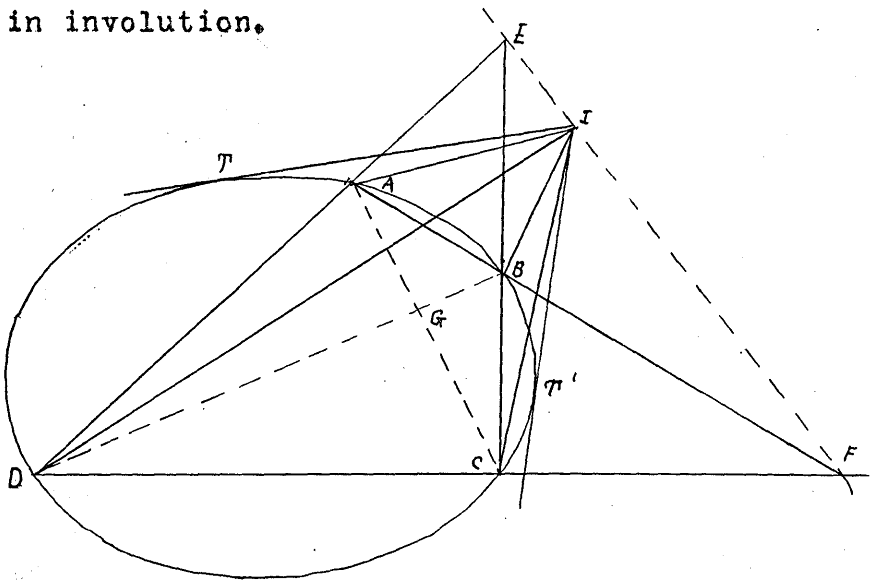
Then S , S' ; P, P' ; Q, Q' are three pairs of points in involution and Q' is uniquely determined as the mate of Q. As the sides of the quadrilateral turn on the three fixed points A^i B^i must always cut 1 in a point conjugate to Q in involution. Since there is only one such point it follows that A^i B^i cuts 1 in a fixed point Q.

NOTE. Compare with the THIRD PROPERTY.

* Taylor. Geometry of Conics. Ex. 743. p. 298.

EIGHTH PROPERTY. GIVEN A QUADRILATERAL INSCRIBED IN A CONIC SUCH THAT THE OPPOSITE SIDES MEET IN E AND F. TANGENTS TO THE CONIC FROM ANY POINT I ON EF, AND THE TWO PAIRS OF LINES JOINING I TO OPPOSITE VERTICES OF THE QUADRILATERAL FORM SIX LINES IN INVOLUTION.*

To prove, IT, IT', IA, IC, ID, IB are six lines in involution.



Proof. The tangents from I are harmonic conjugates with respect to IE and IG. Also the lines IA, IC and IB, ID are harmonic conjugates with respect to IF and IG, since these divide harmonically the diagonals AC and BD. Thus it follows that the three pairs of lines thru I are in involution.

* Chasles, C onic Sections. p. 89.

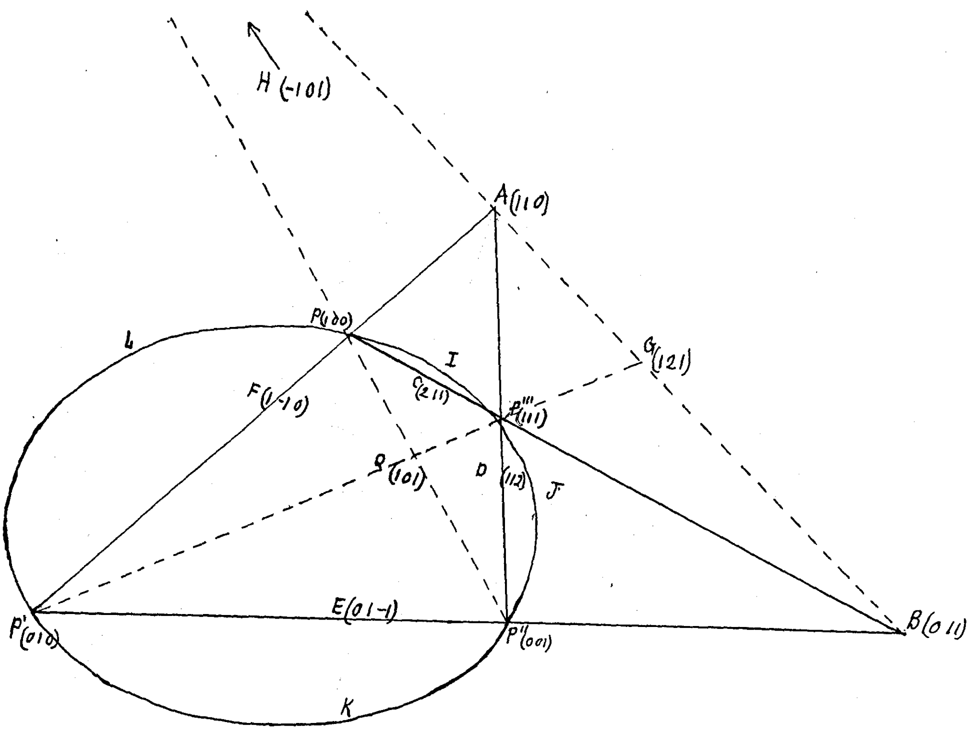
NINTH PROPERTY. IN ANY PARALLELOGRAM INSCRIBED IN A CONIC, THE SIDES ARE PARALLLEL TO TWO CONJUGATE DIAMETERS AND THE DIAGONALS INTERSECT IN THE CENTER.*

Proof. C onsider the parallelogram as a complete quadrangle. Since its three diagonal points must be the vertices of a self conjugate triangle, one of them will be the center of the conic and the other two will be the points at infinity on two conjugate diameters.

§4. ANALYTIC METHOD OF INVESTIGATION.

The purpose of this section is to suggest a simple method for investigating the properties of the inscribed auadrilateral from the purely analytic side. Owing to the fact that *any* inscribed quadrilateral is projectively equivalent to any other inscribed quadrilateral in the same plane, the work may be greatly simplified by the use of projective coordinates.

It is not the purpose to attempt to prove all of the preceding properties, but a few examples are given which show the general method.



Assign the coordinates (1,0,0), (0,1,0), (001) and (1,1,1) to the points P, P', P'', P''' respectively.

Then the equation of the family of conics thru P, P', P'', P''' is

$$x_1 x_2 + a x_1 x_3 - (1 + a) x_2 x_3 = 0 \text{ -----------(1)}$$

Equations of lines in above figure,

P P',	$x_3 = 0$	AB,	$x_1 - x_2 + x_3 = 0$
P P'',	$x_2 = 0$	AQ,	$x_1 - x_2 - x_3 = 0$
P P''',	$x_2 - x_3 = 0$	BQ,	$x_1 + x_2 - x_3 = 0$
P' P'',	$x_1 = 0$		
P' P''',	$x_1 - x_3 = 0$		
P'' P''',	$x_1 - x_2 = 0$		

<u>NOTE</u>. This figure is used for the following examples.

EXAMPLE ONE.

To prove the property that, in the figure, the line LJ is the polar of the point A.

Proof. Equation of line AJ is

$$x_1 = x_2 + x_3(1 + 2a - 2\sqrt{a + a}) \text{ --------(2)}$$

To show AJ tangent to the conic, substitute value of x_1 from (2) in the equation of the conic (1) and

$$x_2^2 + 2(a - \sqrt{a + a^2})x_2 x_3 + (a + 2a^2 - 2a\sqrt{a + a^2})x_3^2 = 0$$

$$[x_2 + (a - \sqrt{a + a^2})]^2 = 0$$

which is the condition that AJ be tangent to the conic at J.

In a similar manner it may be shown that AL is tangent to the curve at L. Therefore from the definition of a polar* it follows that JL is the polar of A.

EXAMPLE TWO.

To prove the property that the tangents to the conic at P and P'', of the figure, meet on the line AB.

Proof. The equations of the tangents to the conic at P and P'' are respectively,

$$x_2 + ax_3 = 0 \text{ -----------------(3)}$$

$$ax_1 - (1 + a)x_2 = 0 \text{ ----------(4)}$$

Coordinates of intersection of (3) and (4) are $-(1 + a)$, $-a$, 1. To show that this point of intersection is on AB it is necessary to substitute its

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* Whitworth, Trilinear Coordinates, p. 233.

coordinates in the equation of AB, $x_1 - x_2 + x_3 = 0$

Substituting, $x_1 - x_2 + x_3 = 0$ becomes,

$$-(1 + a) + a + 1 = 0$$

Since these coordinates satisfy the equation of the line AB, the point is on the line AB.

EXAMPLE THREE.

To prove the property that, in the figure, tangents to the conic at P and P''' meet on the line AQ.

Proof. Equations to the conic at P and P''' are respectively

$x_2 + ax_3 = 0$ ----------------------(3)

$(1 + a)x_1 - ax_2 - x_3 = 0$ -----------(5)

Coordinates of intersection of (3) and (5) are $(1 - a), -a, 1$.

Substitue these coordinates in the equation of the line AQ, $x_1 - x_2 - x_3 = 0$ and have

$$(1 - a) + a - 1 = 0$$

from which it follows that the intersection of tangents to the conic at P and P''' is on the line AQ.

B I B L I O G R A P H Y .

Allman, G. J. London, 1889.

"Greek Geometry from Thales to Euclid."

Catalan, E. Paris, 1879.

"Theoremes et Problemes de Geometrie."

Cayley, Arthur. Cambridge, 1896.

"C ollected Mathematical Papers. Vol. X."

Cantor, Moritz. Leipzig, 1894.

"Vorlesungen uber Geschichte der Mathematik."

Chasles, M. Paris, 1889.

"Apercu Historique des Methodes en Geometrie."

Paris, 1865.

"Traite des Sections Coniques."

C hauvenet, W. Philadelphia, 1888.

"Elementary Geometry."

Cremona, L. Oxford, 1885.

"Elements of Projective Geometry."

Encyklopadie der Mathematischen Wissenschaften,

Band III_2 Heft 1.

F errers, N. M. London, 1866.

"Trilinear C oordinates."

Gow, James. Cambridge, 1884.

"A Short History of Greek Mathematics."

Heath, T. L. Cambridge, 1896.

"Apollonius of Perga."

Cambridge, 1908.

"Thirteen Books of Euclids' Elements." 3 Vols.

Hadamard, J. Paris, 1898.

"Lecons de Geometrie."

Heiberg,

"Archimedes."

Lucklan, R. London, 1893.

"Modern Pure Geometry."

Mulcahy, John. Dublin, 1862.

"Principles of Modern Geometry."

Newton,

"Principia." Lib. I.

Nouvelles Annales, 2 me. Serie, t, X.

Poncelet, Paris, 1865.

"Traite des Proprietes Projectiones." In 2 Vols.

Russel, J. W. Oxford, 1905.

"Elementary Treatise of Pure Geometry."

Salmon, Geo. London, 1869.

"Treatise on Conic Sections."

Steiner, Jacob. Berlin, 1881.

"Gesammelte Werke." Vol. II.

Taylor, Chas. London, 1874.

"Introduction to Ancient and Modern Geometry."

Todhunter, London, 1874.

"Elements of Euclid."

Whitworth, Cambridge, 1866.

"Trilinear Coordinates."